2019-Nov-05 (pre-lecture) Prof. Yun William Yu MATB44

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Outline:

- Linear independence of functions (redux)
- Matrices and systems of ODEs

Recall: A set of functions $f_1, \ldots, f_n$ is **linearly dependent** on an interval I if there exists a set of constants $c_1, \ldots, c_n$ not all 0 s.t.

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in I$. Otherwise, they are **linearly independent**.

This definition can be hard to work with, because for independence you have to show there are no $c_1, \ldots, c_n$, not all 0.

- **Ex.** $e^{px}, \; e^{qx}, \quad p \neq q.$

  Assume they are linearly dependent.

  Then $c_1 e^{px} = c_2 e^{qx}$. WLOG, $c_1 \neq 0$. (since at least one of $c_1 \neq 0$ or $c_2 \neq 0$)

  $$\Rightarrow e^{px} = \frac{c_2}{c_1} e^{qx}$$

  $$\Rightarrow e^{(p-q)x} = \frac{c_2}{c_1}.$$

  This is impossible for any interval larger than a single point because the LHS is variable and the RHS is constant.

  Thus, they must be linearly independent.

- **Ex** What about $x^2 \sin x$, $\sqrt{x} \log x$, e^x, (x^2+5x+1) ?

  No symmetry, so harder to make the same arguments.

  Luckily, we can borrow other techniques from linear algebra.

Define: Given a matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_n \end{bmatrix},$$

the **determinant** $|A|$ is defined by

(1) $|I| = 1 \qquad \left(I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right)$

(2) Exchanging 2 rows flips the sign:

Consider $A = \begin{bmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_n \end{bmatrix}, \quad B = \begin{bmatrix} \vec{B}_1 \\ \vdots \\ \vec{B}_n \end{bmatrix}.$

$$\begin{bmatrix} \vdots \\ \vec{A}_n \end{bmatrix}, \quad \begin{bmatrix} \vdots \\ \vec{B}_n \end{bmatrix}.$$

If $\vec{A}_i = \vec{B}_i$ for all i except two rows p and q,
and $\vec{A}_p = \vec{B}_q$, $\vec{A}_q = \vec{B}_p$, then $|A| = -|B|$.

(3)
$$\begin{vmatrix} \vec{A}_1 \\ \vdots \\ t\vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix} = t\,|A| \quad \text{and} \quad \begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i + \vec{B}_i \\ \vdots \\ \vec{A}_n \end{vmatrix} = \begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix} + \begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{B}_i \\ \vdots \\ \vec{A}_n \end{vmatrix}$$

(i.e. the determinant acts as a linear function on the rows of a matrix).

These properties define the determinant.

e.g.
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= ac\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + ad - bc + bd\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

Note $\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$. same

So $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Lemma: (linear alg.) The following are equivalent:

- $|A| \neq 0$
- rows of A are linearly ind.
- columns of A are linearly ind.
- A has a 0-dimensional nullspace.

(proof by using manipulations above to transform a matrix with linearly dependent rows)

Can we use determinants to say something similar about the linear independence of functions?

Define: The **Wronskian** of a set of functions $f_1(x), \dots, f_n(x) \in C^{n-1}(I)$ is defined to be

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Notation: $f(x) \equiv 0$ means $f(x) = 0 \ \forall x \in I$.

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Theorem (63.55): If $f_1, \dots, f_n \in C^{n-1}(I)$ are linearly dependent on I, then $W(f_1, \dots, f_n) \equiv 0$.

proof.: For some $c_1, \dots, c_n$ not all 0, $\forall x \in I$

$$\left.\begin{aligned} c_1 f_1 + \cdots + c_n f_n &= 0 \\ c_1 f_1' + \cdots + c_n f_n' &= 0 \\ &\vdots \\ c_1 f_1^{(n-1)} + \cdots + c_n f_n^{(n-1)} &= 0 \end{aligned}\right\} \begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

But $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq 0$ so the nullspace of $\begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$ is at least 1-dimensional,

so $W(f_1, \dots, f_n) \equiv 0$. $\square$

Contrapositive: If not $W(f_1, \dots, f_n) \equiv 0$, then $f_1, \dots, f_n$ are linearly independent.

Note: It is NOT true that if $W(f_1, \dots, f_n) \equiv 0$, $f_1, \dots, f_n$ are linearly dependent.

e.g. $x^2, \; |x|x$

$$\begin{vmatrix} x^2 & |x|x \\ 2x & |2x| \end{vmatrix} = x^2|2x| - 2x|x|x = 0 \quad \text{for all } x.$$

But $x^2, |x|x$ are not linearly dependent in any neighborhood of 0.

Theorem 64.11: Let $f_n y^{(n)} + f_{n-1} y^{(n-1)} + \cdots + f_1 y' + f_0 y = 0$, $f_i \in C^0(I)$ and $f_n(x) \neq 0$ for $x \in I$.
If $y_1, \dots, y_n$ are solutions on I, then

$$W(y_1, \dots, y_n) \equiv 0 \quad \text{or} \quad W(y_1, \dots, y_n)(x) \neq 0 \text{ for any } x \in I$$

Theorem 64.13: If $y_1, \dots, y_n$ are solutions on I as above, and if $W(y_1, \dots, y_n)(x_0) = 0$ for some $x_0 \in I$, then $y_1, \dots, y_n$ is linearly dependent on I.

Together, we have a necessary and sufficient condition for independence of solutions to a linear homogeneous ODE.

i.e. Let $y_1, \dots, y_n$ be solutions.

If $y_1, \dots, y_n$ are dependent, then $W(y_1, \dots, y_n) \equiv 0$.

If $W(y_1, \dots, y_n)(x_0) = 0$, for $x_0 \in I$, $W(y_1, \dots, y_n) \equiv 0$

If $W(y_1, \dots, y_n) \equiv 0$, then $y_1, \dots, y_n$ are dependent.

Ex. $\sin x$, $\cos x$, sol. to $y'' + y = 0$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0 \quad \text{for any } x \in \mathbb{R}$$

Ex. e^x, xe^x, e^{2x}, sol to $\left(\frac{d}{dx} - 1\right)^2 \left(\frac{d}{dx} - 2\right) x = 0$

$$\begin{vmatrix} e^x & xe^x & e^{2x} \\ e^x & e^x + xe^x & 2e^{2x} \\ e^x & 2e^x + xe^x & 4e^{2x} \end{vmatrix} = e^x \begin{vmatrix} e^x + xe^x & 2e^{2x} \\ 2e^x + xe^x & 4e^{2x} \end{vmatrix} - xe^x \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} + e^{2x} \begin{vmatrix} e^x & e^x + xe^x \\ e^x & 2e^x + xe^x \end{vmatrix}$$

$$= 4e^{4x} + 4xe^{4x} - \left(4e^{4x} + 2xe^{4x}\right)$$

$$- \left(4xe^{4x} - 2xe^{4x}\right)$$

$$+ \left(2e^{4x} + xe^{4x}\right) - \left(e^{4x} + xe^{4x}\right)$$

$$= e^{4x} \neq 0 \quad \text{for any } x \in \mathbb{R}.$$

Note: even though we now have a formula, it's a pain to work with. Sometimes, it's easier to use vector space properties to prove independence.

Ex. Prove that

$y_1 = xe^x + e^{2x}$, $y_2 = e^x - e^{2x}$, $y_3 = e^x + xe^x + e^{2x}$ are linearly ind.

proof. We know from above that xe^x, e^{2x}, e^x are linearly ind., so $\text{span}\{xe^x, e^{2x}, e^x\}$ is 3-dimensional.

But $e^{2x} = y_3 - y_1 - y_2$

$e^x = y_3 - y_1$

$xe^x = y_1 - e^{2x} = y_1 - (y_3 - y_1 - y_2) = -y_3 + y_2 + 2y_1$

So $\text{span}\{y_1, y_2, y_3\} = \text{span}\{e^x, e^{2x}, xe^x\}$, which is 3 dimensional.

Thus, y_1, y_2, y_3 must be linearly independent.

Linear equations

Consider $\dot{x}(t) = Ax(t)$, $x(0)=x_0$, $A \in M_{n\times n}(\mathbb{R})$ ($n\times n$ matrix), $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Equivalently,

$$\dot{x}_1(t) = A_{1,1}x_1(t) + A_{1,2}x_2(t) + \cdots + A_{1,n}x_n(t) \qquad x_1(0) = x_{0,1}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$\dot{x}_n(t) = A_{n,1}x_1(t) + A_{n,2}x_2(t) + \cdots + A_{n,n}x_n(t) \qquad x_n(0) = x_{0,n}.$$

This is a linear first-order system of ODEs.

Recall that we defined Picard iteration on matrices, so we can get a solution:

$$x_0(t) = x_0$$

$$x_1(t) = x_0 + \int_0^t Ax_0(s)\,ds = x_0 + Ax_0\int_0^t ds = x_0 + tAx_0$$

$$x_2(t) = x_0 + \int_0^t Ax_1(s)\,ds = x_0 + A\int_0^t (x_0 + sAx_0)\,ds = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0$$

$$x_3(t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \frac{t^3}{3!}A^3x_0$$

$$\vdots$$

$$x_m(t) = \sum_{j=0}^{m} \frac{t^j}{j!}A^j x_0$$

$$\vdots$$

$$x(t) = \lim_{m\to\infty} x_m(t) = \sum_{j=0}^{m} \frac{t^j}{j!}A^j x_0.$$

Suppose A is one-dimensional. Then this is a Taylor series for $\exp(tA)x_0$.

Thus, we define the matrix exponential $\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!}A^j$, where $A^0 = I$.

So $x(t) = \exp(tA)x_0$.

A few properties:

- $\exp(0) = I$
- $\exp(X^T) = (\exp X)^T$ $\quad (T = \text{transpose})$
- $\exp(X^*) = (\exp X)^*$ $\quad (* = \text{conjugate transpose})$
- If Y is invertible, $\exp(YXY^{-1}) = Y \exp(X) Y^{-1}$
- If $XY = YX$, $\exp(X+Y) = \exp(X)\exp(Y)$

(note: not generally true)

↳ $\exp((a+b)X) = \exp(aX)\exp(bX)$

↳ $\exp(X)\exp(-X) = I$

- $\det(e^A) = e^{\operatorname{tr}(A)}$
- $\exp\left(\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}\right) = \begin{bmatrix} \exp(d_1) & & 0 \\ & \ddots & \\ 0 & & \exp(d_n) \end{bmatrix}$

Back to ODEs: Given $\dot{x} = Ax$, $x(0) = x_0$

$$x = \exp(tA)x_0 .$$

This should look familiar. Given $\dot{x} = ax$, $x(0) = x_0$ all scalars,

$$\frac{dx}{dt} = ax \Rightarrow \frac{dx}{x} = a\,dt$$

$$\ln x = at + C$$

$$x = Ce^{at}, \quad x(0) = x_0 \Rightarrow C = x_0$$

$$\Rightarrow x = x_0 e^{at}$$

What about a 2nd order ODE: $\ddot{z} + 3\dot{z} + 2z = 0$, $z(0) = 1$, $\dot{z}(0) = 1$

Convert to system of 1st order ODEs

$$\left.\begin{aligned} x_1 &= z \\ x_2 &= \dot{z} \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= \dot{z} = x_2 \\ \dot{x}_2 &= \ddot{z} = -3\dot{z} - 2z = -3x_2 - 2x_1 \end{aligned}$$

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

$$\left.\begin{aligned} x_1(0) &= 1 \\ x_2(0) &= 1 \end{aligned}\right\} \; x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then $$Ax = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \dot{x}.$$

$$\dot{x} = Ax, \qquad x(0) = x_0$$

$$x(t) = \exp(tA)\, x_0$$

Let's change the basis of A to make it nicer to work with.

Eigenvalues of A are computed $\quad Ax = \lambda x$

$$\Rightarrow (\lambda I - A)x = 0$$

$$\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} x = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0$$

Characteristic polynomial $\quad \lambda(\lambda+3) + 2 = \lambda^2 + 3\lambda + 2 = 0.$

$$\lambda = -1, -2.$$

Remember

$\ddot{z} + 3\dot{z} + 2 = 0$

Char poly

$m^2 + 3m + 2 = 0$

Eigenvector corresponding to $\lambda_1 = -1 \quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$

$$\left.\begin{aligned} \Rightarrow \quad x_2 &= -x_1 \\ -2x_1 - 3x_2 &= -x_2 \end{aligned}\right\} \quad v_1 = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \text{ e.g. } v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_2 = -2 \quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}$

$$x_2 = -2x_1 \quad \Rightarrow \quad v_2 = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} \text{ e.g. } v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So we can construct a change of basis matrix

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Then $$P\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}P^{-1} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}. \quad \text{Let } \Lambda = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$$

Then $$x(t) = \exp(tA)x_0 = \exp(tP\Lambda P^{-1})x_0 = P\exp(t\Lambda)P^{-1}x_0$$

$$= P\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}P^{-1}x_0$$

$$= \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}x_0$$

$$= \begin{bmatrix}e^{-t} & e^{-2t}\\ -e^{-t} & -2e^{-2t}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}x_0$$

(Wronshian of $\{2e^{-t}-e^{-2t}, e^{-t}-e^{-2t}\}$)

$$= \begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}3e^{-t} - 2e^{-2t}\\ -3e^{-t}+4e^{-2t}\end{bmatrix} \begin{matrix}\leftarrow z(t)\\ \leftarrow \dot{z}(t)\end{matrix}$$

In the general order 2 case, we have

$$\dot{x} = Ax \qquad x(0) = x_0.$$

If A has 2 distinct eigenvalues λ_1, λ_2, then

$$A = P\Lambda P^{-1}, \quad \text{where} \quad \Lambda = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}.$$

We can change coordinates to

$$y(t) = P^{-1}x(t), \qquad y(0) = y_0 = P^{-1}x_0.$$

In this new basis, we have $\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}P\Lambda P^{-1}x = \Lambda y$, $y(0)=y_0$

The solution is $$y(t) = \exp(t\Lambda)y_0 = \begin{pmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}y_0$$

We can convert back to the original basis by

$$x(t) = P\exp(t\Lambda)P^{-1}x_0 = P\begin{pmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}P^{-1}x_0.$$

$$x(t) = P\exp(t\Lambda)P^{-1}x_0 = P\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix}P^{-1}x_0 .$$

But the solution is easier to interpret in the eigenbasis ← Next time